Math 105 - Assignment 3: Solutions

1. We have that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$

Since we are asked to use midpoint rule we have

$$x_i^* = \frac{x_{i-1} + x_i}{2}$$

$$= \frac{a + (i-1)\Delta x + a + i\Delta x}{2}$$

$$= \frac{2a + (2i-1)\Delta x}{2}$$

$$= a + (i - \tfrac{1}{2})\Delta x$$

Now $\int_0^6 x^2\,dx$, we have $a=0$, $b=6$, $\Delta x = \frac{6}{n}$, $x_i^* = (i-\frac{1}{2})\frac{6}{n}$.

So $\int_0^6 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i^*)^2 \Delta x$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[(i-\tfrac{1}{2})\tfrac{6}{n}\right]^2 \frac{6}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (i-\tfrac{1}{2})^2 \frac{6^3}{n^3}$$

$$= \lim_{n\to\infty} \frac{6^3}{n^3} \sum_{i=1}^{n} (i-\tfrac{1}{2})^2$$

$$= \lim_{n\to\infty} \frac{6^3}{n^3} \sum_{i=1}^{n} i^2 - i + \tfrac{1}{4}$$

$$\int_0^b x^2 dx = \lim_{n\to\infty} \frac{b^3}{n^3}\sum_{i=1}^{n} i^2 - \frac{b^3}{n^3}\sum_{i=1}^{n} i + \frac{b^3}{4n^3}\sum_{i=1}^{n} 1$$, by linearity

$$= \lim_{n\to\infty} \frac{b^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{b^3}{n^3}\frac{n(n+1)}{2} + \frac{b^3}{4n^3}n$$, by formulas for $\sum_{i=1}^{n} 1, \sum_{i=1}^{n} i, \sum_{i=1}^{n} i^2$ from class.

$$= \lim_{n\to\infty} \frac{b^3}{6}\left(\frac{2n^3+3n^2+n}{n^3}\right) - \frac{b^3}{2}\left(\frac{n^2+n}{n^3}\right) + \frac{b^3}{4n^2}$$

$$= \lim_{n\to\infty} \frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) - \frac{b^3}{2}\left(\frac{1}{n}+\frac{1}{n^2}\right) + \frac{b^3}{4n^2}$$

$$= \frac{2b^3}{6}$$

$$= \frac{b^3}{3}$$

2. We want to evaluate:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i e^{\frac{2i}{n}}}{n^2} =: L$$

Recall $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, $x_i = a + i\Delta x$, $\Delta x = \frac{b-a}{n}$

So $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\frac{b-a}{n}$$

So we want to find, a, b, f such that

$$f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} = \frac{i e^{\frac{2i}{n}}}{n^2} \quad (\star)$$

Note that there is no sum on the right hand side of $(\star)$ so $a=0$.

Now we still need to find b and f. Since $a=0$, $\Delta x = \frac{b}{n}$ and $(\star)$ becomes

$$f\left(i\frac{b}{n}\right)\frac{b}{n} = \frac{i e^{\frac{2i}{n}}}{n^2} \quad (\star\star)$$

Since on the left hand side we always have "i" and "$\frac{b}{n}$" together, we must have the same in the right hand side of $(\star\star)$. Thus there are 2 choices for b that make sense, $b=2$ or $b=1$. Let us find f in both cases.

Case (1) $b = 2$, (★★) tells us

$$f\left(\frac{2i}{n}\right)\frac{2}{n} = \frac{i e^{\frac{2i}{n}}}{n^2}$$

$$\Rightarrow f\left(\frac{2i}{n}\right) = \frac{i}{2n} e^{\frac{2i}{n}}$$

Let $x = \frac{2i}{n}$, so $i = \frac{nx}{2}$, thus

$$f(x) = \frac{\left(\frac{nx}{2}\right) e^{\frac{2\left(\frac{nx}{2}\right)}{n}}}{2n}$$

$$= \frac{xe^x}{4}$$

Thus $a = 0$, $b = 2$, $f(x) = \frac{xe^x}{4}$, and

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2} e^{\frac{2i}{n}} = \frac{1}{4}\int_0^2 xe^x \, dx$$; $u = x$, $dv = e^x dx$, $du = dx$, $v = e^x$

$$= \frac{1}{4}\left[xe^x\Big|_0^2 - \int_0^2 e^x dx\right]$$

$$= \frac{1}{4}\left[2e^2 - (e^2 - e^0)\right]$$

$$= \frac{1}{4}(e^2 + 1)$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

Case ② $b=1$. (☆☆) tells us

$$f\left(\frac{i}{n}\right)\frac{1}{n} = \frac{i e^{\frac{2i}{n}}}{n^2}$$

$$\Rightarrow \quad f\left(\frac{i}{n}\right) = \frac{i e^{\frac{2i}{n}}}{n}$$

Let $x=\frac{i}{n}$, so $i = nx$, thus

$$f(x) = \frac{nx\, e^{\frac{2nx}{n}}}{n}$$

$$= x e^{2x}$$

Thus $a=0$, $b=1$, $f(x) = xe^{2x}$, and

$$\lim_{n\to\infty}\sum_{i=1}^{n} \frac{i}{n^2} e^{\frac{2i}{n}} = \int_0^1 xe^{2x}\,dx \quad , \quad u=x,\ dv=e^{2x}dx$$

$$du=dx,\ v=\frac{1}{2}e^{2x}$$

$$= \frac{1}{2}xe^{2x}\Big|_0^1 - \int_0^1 \frac{e^{2x}}{2}\,dx$$

$$= \frac{e^2}{2} - \left[\frac{e^{2x}}{4}\right]_0^1$$

$$= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

Note that these 2 case seem very different, they are essentially the same, since

$$\int_0^1 xe^{2x}\,dx, \quad \text{Let } t=2x,\ dt=2dt$$

$$\Rightarrow \quad x=\frac{t}{2},\ dx=\frac{dt}{2}$$

$$= \int_0^2 \frac{t}{2}e^t\frac{dt}{2}$$

$$= \int_0^2 \frac{1}{4}te^t\,dt$$

Thus the 2 choices we made were mearly a substitution in disguise! ⑤

3. Let $g(x) = \int_{\log x}^{e^{x^2}} \log(t^2)\,dt$

Recall fundamental theorem of Calculus I say if

$$F(x) = \int_a^x f(t)\,dt$$

Then $F'(x) = f(x)$

Let $F(x) = \int_a^x \log(t^2)\,dt$, So $F'(x) = \log(x^2)$

Now $g(x) = \int_{\log x}^{e^{x^2}} \log(t^2)\,dt$

$$= \int_a^{e^{x^2}} \log(t^2)\,dt - \int_a^{\log x} \log(t^2)\,dt$$

$$= F(e^{x^2}) - F(\log x)$$

Now by chain rule

$$g'(x) = F'(e^{x^2})(e^{x^2})' - F'(\log x)(\log x)'$$

$$= F'(e^{x^2})\, e^{x^2} \cdot 2x - F'(\log x)\,\frac{1}{x}$$

$$= \log((e^{x^2})^2) \cdot 2xe^{x^2} - \log((\log x)^2)\,\frac{1}{x}$$

$$= 2xe^{x^2}\log(e^{2x^2}) - \frac{1}{x}\log((\log x)^2)$$

$$= 2xe^{x^2}\,2x^2 - \frac{2}{x}\log(\log(x))$$

$$= 4x^3e^{x^2} - \frac{2}{x}\log(\log(x))$$

4. Recall a function $f(x)$ is increasing when $f'(x) > 0$.

$$f(x) = \int_0^{x^2} \frac{1-t}{1+t}\, dt$$

So we need to find when $f'(x) > 0$.

$$f'(x) = \frac{1-x^2}{1+x^2} \cdot 2x \text{ , by FTC I}$$

$$= \frac{2x(1-x)(1+x)}{1+x^2}$$

Now the denominator is alway positive so the critical points of f are when the numerator is 0. Thus the critical points are

$$x = -1, 0, 1$$

Now lets use the first derivative test, by picking test points in the intervals: eg $x = -2, -\frac{1}{2}, \frac{1}{2}, 2$ to get:

$f'(x)$	$+$		$-$		$+$		$-$
		-1		0		1	

Thus $f'(x) > 0$ when $x < -1$, and $0 < x < 1$.

thus $f(x)$ is increasing when $x \in (-\infty, 1) \cup (0, 1)$.

5. Note, there may be more than one way to do some of these, but they all should produce the same answer.

a) $\int \frac{x}{x+1}\,dx$, $\quad u = x+1$, $du = dx$

$x = u-1$

$= \int \frac{u-1}{u}\,du$

$= \int 1 - \frac{1}{u}\,du$

$= u - \log|u| + c$

$= x+1 - \log|x+1| + c$

b) $\int x^2 (x+1)^{3/2}\,dx$, $\quad u = x+1$, $du = dx$

$x = u-1$

$= \int (u-1)^2 u^{3/2}\,du$

$= \int (u^2 - 2u + 1) u^{3/2}\,du$

$= \int u^{\frac{7}{2}} - 2u^{5/2} + u^{3/2}\,du$

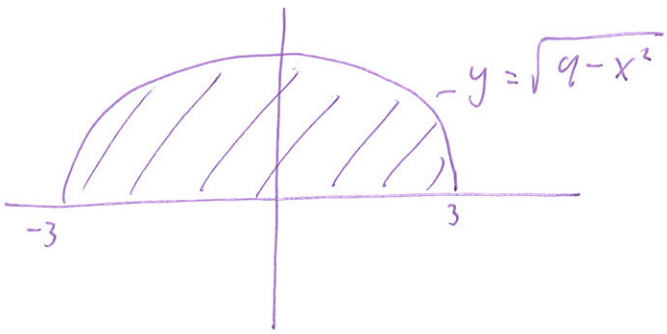
$= \frac{2}{9} u^{\frac{9}{2}} - \frac{4}{7} u^{\frac{7}{2}} + \frac{2}{5} u^{5/2} + c$

$= \frac{2}{9}(x+1)^{\frac{9}{2}} - \frac{4}{7}(x+1)^{\frac{7}{2}} + \frac{2}{5}(x+1)^{5/2} + c$

Alternatively you could have used integration by parts twice.

c) $\int_{-3}^{3} \sqrt{9-x^2}\, dx$

Note that $y=\sqrt{9-x^2}$ is the equation of the upper half of a circle of radius 3 centered at (0.0).



So the area of the shaded region represents $\int_{-3}^{3} \sqrt{9-x^2}\, dx$

Since the area of a circle is πr^2, we have the area of the semi-circle is $\frac{\pi r^2}{2}$. Therefore:

$$\int_{-3}^{3} \sqrt{9-x^2}\, dx = \frac{\pi(3)^2}{2}$$
$$= \frac{9\pi}{2}$$

d) Let $f(x) = \sin(x^{2015}) - x^{2015}$

$$f(-x) = \sin((-x)^{2015}) - (-x)^{2015}$$
$$= \sin(-x^{2015}) + x^{2015}, \text{ since } x^{2015} \text{ is odd}$$
$$= -\sin(x^{2015}) + x^{2015}, \text{ since } \sin x \text{ is odd}$$
$$= -[\sin(x^{2015}) + x^{2015}]$$
$$= -f(x).$$

Thus $f(x)$ is odd. Since $\int_{-a}^{a} g(x)\,dx = 0$ if g is odd,

we have
$$\int_{-2015}^{2015} \sin(x^{2015}) - x^{2015}\,dx = 0$$

e) $\int (\log x)^2\,dx$, $u = (\log x)^2$, $dv = dx$

$du = \frac{2\log x}{x}\,dx$, $v = x$

$= x(\log x)^2 - \int 2\frac{\log x}{x} \cdot x\,dx$

$= x(\log x)^2 - 2\int \log x\,dx$, $u = \log x$, $dv = dx$

$du = \frac{1}{x}\,dx$, $v = x$

$= x(\log x)^2 - 2\left[x\log x - \int \frac{1}{x} \cdot x\,dx\right]$

$= x(\log x)^2 - 2x\log x + 2\int dx$

$= x(\log x)^2 - 2x\log x + 2x + C$

f) $\int_0^{\pi^2} \sin(\sqrt{x})\,dx$, $t = \sqrt{x}$, $t(0) = 0$, $t(\pi^2) = \pi$

$dt = \frac{1}{2\sqrt{x}}\,dx \Rightarrow dx = 2\sqrt{x}\,dt$

$= 2t\,dt$

$= \int_0^{\pi} 2t\sin t\,dt$, $u = 2t$, $dv = \sin t$

$du = 2dt$ $v = -\cos t$

$= -2t\cos t\Big|_0^{\pi} - \int_0^{\pi} -2\cos t\,dt$

$= -2\pi\cos\pi + 2\int_0^{\pi}\cos t\,dt$

$= 2\pi + 2[\sin t]_0^{\pi}$

$= 2\pi + 2\sin\pi - 2\sin 0$

$= 2\pi$

y) $\int_0^t e^s \cos(t-s)\,dx$, $u = e^s$, $dv = \cos(t-s)\,ds$

$du = e^s ds$ $\quad v = -\sin(t-s)$

$= -e^s \sin(t-s)\Big|_0^t - \int_0^t -e^s \sin(t-s)\,ds$

$= -e^t \sin(t-t) - (-e^0 \sin(t-0)) + \int_0^t e^s \sin(t-s)\,ds$

$= -e^t \sin(0) + e^0 \sin t + \int_0^t e^s \sin(t-s)\,ds$

$= \sin t + \int_0^t e^s \sin(t-s)\,ds$, $u = e^s$, $dv = \sin(t-s)\,ds$

$du = e^s ds$ $\quad v = \cos(t-s)$

$= \sin t + \Big[e^s \cos(t-s)\Big]_0^t - \int_0^t e^s \cos(t-s)\,ds$

$= \sin t + e^t \cos 0 - e^0 \cos t - \int_0^t e^s \cos(t-s)\,ds$

$= \sin t + e^t - \cos t - \int_0^t e^s \cos(t-s)\,ds$

$\Rightarrow \int_0^t e^s \cos(t-s)\,ds = \sin t + e^t - \cos t - \int_0^t e^s \cos(t-s)\,ds$

$\Rightarrow 2\int_0^t e^s \cos(t-s)\,ds = \sin t + e^t - \cos t$

$\Rightarrow \int_0^t e^s \cos(t-s)\,ds = \dfrac{\sin t + e^t - \cos t}{2}$